On Unit Root Tests in the Presence of Transitional Growth

by

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Abstract:

Standard unit root tests do not account for \( \beta \)-convergence. Consequently, augmented Dickey-Fuller (ADF) tests have incorrect size when applied to series with transitional growth dynamics. We illustrate this issue with German GDP data from 1950 onwards. Recently advanced atheoretic linear trend models fail to take the post World-War II catch-up process properly into account. We use the Solow growth model to discriminate between transitional catch-up dynamics and long-run equilibrium growth. An appropriate unit root test is due to Lanne, Lütkepohl and Saikkonen (2002). Simulation results indicate that this test has correct size, while ADF-tests are biased towards rejecting the null hypothesis too often.

Keywords: Solow growth model, transitional dynamics, unit root tests

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1. Introduction

Tests of the unit root hypothesis are an essential part of empirical macroeconomic research. Similarly, tests of the convergence hypothesis are an essential part of the empirical growth literature. Both types of tests reject their null hypothesis (a unit root or no convergence, respectively) if there is significant evidence for a negative correlation between the first difference of a variable and its lagged level. But the tests test for completely different phenomena: Trend stationary versus difference stationary representations of a variable relate to the nature of its long-run growth path, while convergence is an issue of transitional growth. Since the basic principle of the tests is the same, size distortions may be expected when tests for unit roots are conducted in the presence of transitional growth.

This issue is of obvious importance. Modern time series models are increasingly used in Eastern European universities, research institutes or central banks for policy simulations or forecasting. But the (still short) time series are typically characterized by post-communist transitional growth dynamics. (More transitional growth impulses are likely to occur after accession of many of these countries to the European Union). Hence, standard unit root tests may be biased towards rejection of the null and may thus induce inappropriate model specification and erroneous results in forecasting and policy analysis.

We illustrate the issue with reference to the German post World-War II growth experience. Recently published studies by Assenmacher (1998) and Meier (2001) have concluded that log GDP (henceforth lnGDP) is trend-stationary. This inference is based on standard Augmented Dickey-Fuller tests with linear trends or on broken linear trends in the framework of Perron (1989). For instance, for yearly data from 1950 to 1998 we obtain (t-statistics in parentheses):

\[
\Delta \ln GDP = 0.71 - 0.098 \ln GDP_{t-1} + 0.268 \Delta \ln GDP_{t-1} - 0.299 \Delta \ln GDP_{t-2} + 0.002 t + \epsilon_t, \tag{1}
\]

\begin{align*}
(3.95) & & (-3.65) & & (1.94) & & (-2.19) & & (2.15)
\end{align*}

The t-statistic of –3.65 for the coefficient of the lagged level is significant at the 5% level of the Dickey-Fuller distribution and it seems as if this justifies a trend stationary model for lnGDP.

Note, however, that such a test does not take into account possible $\beta$-convergence. Hence the significance of the lagged level in (1) could be due to a misspecified model and need not suggest the absence of a stochastic trend. A casual inspection of the German GDP series reveals that the level of GDP is well represented by a linear trend, but the logarithm seems to follow a strictly concave trend, cf. Figure 1. This is due to the fact that growth rates of German GDP have been very high after World War II and have settled to lower levels since. We therefore suggest to view German GDP growth in terms of the neoclassical growth model, where much of the post-World-War-II growth would be interpreted as convergence to a steady-state growth path from an initial position characterized by severe war damages.

We specify the Solow growth model with Cobb-Douglas production function in order to derive an analytic expression for the time path of GDP in Sec. II. (This specification results in Bernoulli’s differential equation whose solution is well known). We then test the unit root hypothesis for German GDP allowing for a proper trend specification in Sec. III. Sec. IV concludes.
II. Convergence to a Balanced Growth Path

In order to find an analytic expression for the growth path of an economy which has lost a large part of its capital stock, let us consider the capital accumulation equation of the Solow growth model for the special case of Cobb-Douglas production. Under constant returns we can write this equation in intensive form as

\[ \dot{k} = sk^\alpha - (n + \delta + \gamma)k \]  

(2)

where \( k = K/AL \) is capital in efficiency units, \( \alpha \) is the capital share, labor \( L \) grows with constant rate \( n \), exogenous factor productivity \( A \) grows with constant rate \( \gamma \) and \( \delta \) is the constant rate of depreciation. This equation is Bernoulli’s differential equation, which can be transformed to linearity by substituting \( z = k^{1-\alpha} \). The general solution in \( z \) as a function of time \( t \) is given by

\[ z = c \exp \left\{ - (1 - \alpha)(n + \delta + \gamma)t \right\} + s(n + \delta + \gamma)^{-1}, \]  

(3)

where \( c \) is an arbitrary constant. Output in intensive form is given by \( Y/AL = y = k^\alpha \), hence we have

\[ \ln y = \alpha(1-\alpha)^{-1} \ln \left[ c \exp \left\{ - (1 - \alpha)(n + \delta + \gamma)t \right\} + s(n + \delta + \gamma)^{-1} \right] \]  

(4)

For a given starting value \( y_0 \) at time \( t = 0 \) we compute \( c = y_0^{(1-\alpha)/\alpha} - s/(n + \delta + \gamma) \). We further have \( y^* := \lim_{t \to \infty} y = \left( s/(n + \delta + \gamma) \right)^{\alpha/(1-\alpha)} \), so that \( c = y_0^{\alpha/(1-\alpha)} - y^* \). i.e. \( c < 0 \) for \( y_0 < y^* \).

Using \( \ln Y = \ln y + \ln(AL) = \ln y + \ln(A_0L_0) + (n + \gamma)t \) and imposing \( c < 0 \) for Germany in the 1950s we have

\[ \ln Y = c_1 + c_2 t + \alpha(1-\alpha)^{-1} \ln \left[ c_3 - \exp \left\{ - (1 - \alpha)(n + \delta + \gamma)t \right\} \right] \]  

(5)

where \( c_1 := \ln(A_0L_0) + \alpha(1-\alpha)^{-1} \ln(-c) \) is arbitrary and \( c_2 := n + \gamma \) and \( c_3 := s/\kappa(n + \delta + \gamma) \) are positive constants. Observe that \( c_1 > 1 \) so that the logarithm in the last term is well defined. While standard unit root tests allow for a constant and a linear trend under the alternative, expression (5) shows that this routine specification neglects a term of the form \( S(t) = \alpha(1-\alpha)^{-1} \ln \left[ c_3 - \exp \left\{ - (1 - \alpha)(c_2 + \delta)t \right\} \right] \) for economies whose GDP is characterized by Solovian transitional dynamics. In the next section we will consider unit root tests which allow for a deterministic term similar to \( S(t) \).
III. Empirical Implementation

Equation (5) describes the time path of log output in a deterministic Solow-model converging to the steady state growth path. For our stochastic model, augmenting the equation by an error term $u_t$ gives

$$y_t = c_t + c_t t + S_t + u_t,$$

where $u_t$ is assumed to be an AR($p$) process, $\alpha^*(B)(1 - \rho B)u_t = \varepsilon_t$ with $\alpha^*(B) = 1 - \alpha_1 B - ... - \alpha_p B^{p-1}$ and $B$ is the backshift operator. Clearly, $y_t$ has a unit root if $u_t$ has one, that is, if $\rho = 1$. Unit root tests for the model (6) with quite general functions $S_t$ (the discrete analogue of $S(t)$ in the Sec. II) have been proposed by Saikkonen and Lütkepohl (2001) and Lanne, Lütkepohl and Saikkonen (2002) (henceforth LLS). These authors allow for functions $S_t$ describing a smooth shift from a given level of the trend function to a new one. For example, the function may be of the type $S_t = \xi f_t(\theta)$, where

$$f_t(\theta) = 1 - \exp\{-\theta t\}.$$

Here $\xi$ and $\theta > 0$ are scalar parameters to be estimated. In fact, Saikkonen and Lütkepohl (2001) and LLS are primarily interested in situations where a structural level shift occurs somewhere in the sample period and therefore they use a slightly more general formulation which allows the shift to begin at some prespecified time in the sample period. For our purposes it is sufficient to use the function specified in (7) because it can describe shifts very much like the one obtained in equation (5). Notice that the function $S(t)$ specified in (5) is overparameterized for estimation purposes. It is easy to see, however, that the actual functional forms that can be generated by (5) are very close to the exponential function specified in (7).

LLS propose estimating the deterministic term in (6) first by a generalized least squares procedure and subtracting it from the original series. Then an ADF type test is performed on the adjusted series. LLS consider different test versions. We use $\tau_{\text{int}}^*$ which is recommended by LLS because of its superior small sample performance. The asymptotic distribution of the test statistic is different from the usual ADF distribution and it is also a nonstandard distribution for which critical values are tabulated in LLS.

Applying this test to the lnGDP series with one lagged difference (sample 1952-1998) gives a value of the test statistic of $\tau_{\text{int}}^* = 0.013$ and the corresponding critical values for sample size $T=50$ from LLS, Table II (see (4.9)), are: -3.81 (1%), -3.15 (5%), -2.86 (10%). As in an ADF test, the null hypothesis is rejected for values of the test statistic smaller than the critical values corresponding to the selected significance level. Consequently, the present test clearly cannot reject the unit root at any common significance level.

The actual estimators of the parameters of the shift term are not easy to interpret and, in fact, $\theta$ and $\xi$ cannot even be estimated consistently in general (see Saikkonen and Lütkepohl (2001), Lemma 1). LLS show that still a useful test distribution is obtained which does not depend on nuisance parameters under the unit root null hypothesis. Despite the possibly poor estimation of $\theta$ and $\xi$, it may be instructive to compare the estimated deterministic term with the actual observations. This comparison can be made in the first panel of Fig. 1, where the estimated deterministic term is represented by a dashed line. Obviously, our estimated shift function captures the catch-up process at the beginning of the sample quite well.
In the second panel of Fig. 1 the estimated $\hat{f}_i(\hat{\theta})$ is shown. Clearly, the catch-up process was not completed at the end of the 1950s according to our estimated function. Of course, given the uncertainty in the parameter estimators, it is difficult to draw precise conclusions regarding the duration of the catch-up process. However, from our estimates it may have lasted well into the sixties.

**Figure 1**

*Unit root test with exponential shift*

To compare our testing procedure with the standard approach, we simulated (6) with calibrated parameters ($\alpha = 0.3$, $\delta = 0.05$, $c_2 = 0.02$, $c_3 = 1.3$, $\alpha = 0.3$, $\rho = 1$, $\sigma = 0.02$). Generating 50 observations and specifying the third observation as the break point (the first two observations are lost due to one lagged difference in the test equation), we perform the $\tau_{\text{int}}$-test and a standard ADF-test with two lags. In 10,000 replications the empirical 5%-quantile of the $\tau_{\text{int}}$-statistic is –3.12. This is almost the same as the simulated 5%-critical value for fifty observations in LLS, -3.15. Conversely, using this critical value, the size of the $\tau_{\text{int}}$-statistic at the nominal 5% level is 4.7%. Simulation results for the ADF-statistic, on the other hand, yield a size of 14.6% at the nominal 5% level. Thus, as expected, the ADF-test is clearly anti-conservative in settings of transitional neoclassical growth.

**IV. Conclusions**

It is the central idea of this paper that catch-up processes must be properly taken into account if unit-root tests are applied to series characterized by transitional growth dynamics. The apparent problem inherent in unit root inference applied to the transitional dynamics of a Solovian growth process lies in the fact that it is difficult to distinguish convergence to a balanced growth path from (persistent or mean-reverting) fluctuations in the neighborhood of this balanced growth path. For the case of a Cobb-Douglas production function, it is possible to solve the Bernoulli differential equation which is at the heart of the Solow model and derive the appropriate correction for transitional growth.

For German GDP, the unit root hypothesis can be tested under a correct specification of the underlying trend. Using a test proposed by LLS, we find no convincing evidence against the unit root null hypothesis. In particular, extending the span of the data over almost half a century results in test statistics which are clearly insufficient to reject the null. This is polar to some recent research which argued that inclusion of GDP data from the 1950s might provide the critical mass of evidence which allows for a rejection of the null. Clearly, the results presented in this paper suggest that such claims should be cautiously reviewed.
importantly, both theoretical insights and simulation results indicate that standard ADF-tests do not have correct size in the presence of transitional growth and are biased towards rejection of the null hypothesis. This finding should be particularly important for transition economies.
References


